Modeling $1/f$ noise

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The noise of signals or currents consisting of a sequence of pulses, elementary events, or moving discrete objects (particles), is analyzed. A simple analytically solvable model is investigated in detail both analytically and numerically. It is shown that $1/f$ noise may result from the statistics of the pulses' transit times, with random increments of the time intervals between the pulses. The model also serves as a basis for revealing parameter dependences of $1/f$ noise, and allows one to make some generalizations. As a result the intensity of $1/f$ noise is expressed through the distribution and characteristic functions of the time intervals between the subsequent transit times of the pulses. The conclusion that $1/f$ noise may result from the clustering of the signal pulses, elementary events, or particles can be drawn from an analysis of the model systems.
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I. INTRODUCTION

The omnipresence of $1/f$ noise is one of the oldest puzzles in contemporary physics. During more than 70 years since its first observation by Johnson, long-memory processes with long-term correlations have been observed in many types of systems from physics, technology, biology, astrophysics, geophysics, and sociology (see Refs. [1–4] and references herein). Recently $1/f$ noise was discovered in human cognition [5], human coordination [6], and even in the distribution of prime numbers [7].

Despite the widespread occurrence of fluctuations of signals and variables exhibiting $1/f^{\delta}$ ($\delta \simeq 1$) behavior of the power spectral density $S(f)$ at low frequencies in a large diversity of systems, no generally recognized explanation of the ubiquity of $1/f$ noise has been proposed. Physical models of $1/f$ noise in some physical systems are usually very specialized, or complicated (see Refs. [1–4] and references herein), and they do not explain the omnipresence of the processes with a $1/f^{\delta}$ spectrum [8–10]. Note also some mathematical analyses [11], models, and algorithms for the generation of processes with $1/f$ noise [12–14]. These models also expose some shortcomings: they are very specific, formal (like ''fractional Brownian motion'' or the half-integral of a white noise signal), or unphysical. They cannot, as a rule, be solved analytically, and they do not reveal the origin nor the necessary and sufficient conditions for the appearance of $1/f$-type fluctuations.

In such a situation the simple analytically solvable model system generating $1/f$ noise may essentially influence the revelation of the origin and essence of the effect. Here we present a model which generates $1/f$ noise in any desirably wide range of frequency. Our model is a result of the search for necessary and sufficient conditions for the appearance of $1/f$ fluctuations in simple systems affected by the random external perturbations, which was initiated in Ref. [15] and originated from the observation of the transition from chaotic to nonchaotic behavior in the ensemble of randomly driven systems [16]. Contrary to the McWhorter model [17], based on the superposition of a large number of Lorentzian spectra and requiring a very wide distribution of relaxation times, our model contains only one relaxation rate γ, and can have an exact $1/f$ spectrum in any desirably wide range of frequency. The model may be used as a basis for checking assumptions made in the derivation of $1/f$ noise spectra for different systems. Furthermore, it allows us to make a heuristic presumption of the generalizations of the theory of $1/f$ noise. Numerical simulations and comparisons with analytical results confirm this supposition.

II. MODEL

In many cases, the intensity of some signal or current can be represented by a sequence of random (however, as a rule, mutually correlated) pulses or elementary events $A_k(t-t_k)$. Here the function $A_k(\varphi)$ represents the shape of the k pulse having an influence on the signal $I(t)$ in the region of transit time t_k. The signal or intensity of the current of particles in some space cross section may, therefore, be expressed as

$$I(t)=\sum_k A_k(t-t_k). \tag{1}$$

It is easy to show that the shapes of the pulses mainly influence the high frequency, $f \geqslant \Delta t_p$ with Δt_p being the characteristic pulse length, power spectral density, while fluctuations of the pulse amplitudes result, as a rule, in white or Lorentzian, but not $1/f$, noise [18]. Therefore, we restrict our analysis to noise due to correlations between the transit times t_k. In such an approach we can replace the function $A_k(t-t_k)$ by the Dirac delta function $\delta(t-t_k)$, and the signal is expressed as

$$I(t)=\sum_k \delta(t-t_k). \tag{2}$$

This model also corresponds to the flow of identical objects: electrons, photons, cars, and so on. On the other hand, fluctuations of the amplitudes A_k may result in additional noise but cannot reduce the $1/f$ noise we are looking for.

The power spectral density of current (2) is

$$S(f)=\lim_{T\to\infty}\left\langle \frac{2}{T}\left|\sum_{k=k_{\min}}^{k_{\max}} e^{-i2\pi f t_k}\right|^2\right\rangle = \lim_{T\to\infty}\left\langle \frac{2}{T}\sum_k \sum_{q=k_{\min}-k}^{k_{\max}-k} e^{i2\pi f(t_{k+q}-t_k)}\right\rangle, \qquad (3)$$

where T is the whole observation time interval, $k_{\min}$ and $k_{\max}$ are minimal and maximal values of index k in the interval of observation, and the brackets $\langle\cdots\rangle$ denote the averaging over realizations of the process.

In this approach the power spectral density of the signal depends on the statistics and correlations of the transit times t_k only. It is well known that a sequence of random, Poisson transit times generates white (shot) noise [18]. The sequence of transit times t_k with random increments $t_k=t_{k-1}+\bar{\tau}+\sigma\varepsilon_k$ (where $\bar{\tau}$ is the average time interval between pulses, $\{\varepsilon_k\}$ denotes the sequence of uncorrelated normally distributed random variables with zero expectation and unit variance, i.e., the white noise source, and σ is the standard deviation of white noise) results in a Lorentzian spectra [15]. Here we will consider sequences of transit times with random increments of the time intervals between pulses, $\tau_k=\tau_{k-1}+\sigma\varepsilon_k$, where $\tau_k=t_k-t_{k-1}$. It is natural to restrict in some way the infinite Brownian increase or decrease of the intervals τ_k, e.g., by the introduction of relaxation to the average period $\bar{\tau}$ rate γ. So we have recurrent equations for the transit times:

$$t_k=t_{k-1}+\tau_k,$$
$$\tau_k=\tau_{k-1}-\gamma(\tau_{k-1}-\bar{\tau})+\sigma\varepsilon_k. \qquad (4)$$

The simplest physical interpretation of model (4) corresponds to one particle moving in a closed contour with the period of the drift of the particle around the contour fluctuating (due to the external random perturbations) about the average value $\bar{\tau}$ [19].

III. SOLUTIONS

An advantage of model (4) is that it may be solved analytically. So, an iterative solution of Eqs. (4) results in an expression for the period,

$$\tau_k=\bar{\tau}+(\tau_0-\bar{\tau})(1-\gamma)^k+\sigma\sum_{j=1}^{k}(1-\gamma)^{k-j}\varepsilon_j, \qquad (5)$$

where τ_0 is the initial period. The dispersion of the period τ_k is

$$\sigma_\tau^2(k)\equiv\langle\tau_k^2\rangle-\langle\tau_k\rangle^2=\frac{\sigma^2[1-(1-\gamma)^{2k}]}{2\gamma(1-\gamma/2)} \simeq \begin{cases}\sigma^2 k, & 2k\gamma\ll 1\\ \sigma^2/2\gamma, & 2k\gamma\gg 1.\end{cases} \qquad (6)$$

Therefore, after a characteristic transition to the stationary process time, $t_{\mathrm{tr}}=\bar{\tau}/\gamma$, the dispersion of the period approaches the limiting value $\sigma_\tau^2=\sigma^2/2\gamma$.

After some algebra we can also obtain an explicit expression for the transit times t_k $(k\geqslant 1)$,

$$t_k=t_0+k\bar{\tau}+\frac{1-\gamma}{\gamma}[1-(1-\gamma)^k](\tau_0-\bar{\tau}) + \frac{\sigma}{\gamma}\sum_{l=1}^{k}[1-(1-\gamma)^{k+1-l}]\varepsilon_l, \qquad (7)$$

where t_0 is the initial time. The dispersion of the transit time t_k is

$$\sigma_t^2(k)\equiv\langle t_k^2\rangle-\langle t_k\rangle^2=\frac{\sigma^2}{\gamma^2}\left\{k-2\frac{1-\gamma}{\gamma}[1-(1-\gamma)^k]+(1-\gamma)^2\frac{1-(1-\gamma)^{2k}}{1-(1-\gamma)^2}\right\}=\begin{cases}\sigma^2(k/6+k^2/2+k^3/3+\cdots), & 2\gamma k\ll 1\\ (\sigma/\gamma)^2(k-3/(2\gamma)+5/4\pm\cdots), & 2\gamma k\gg 1.\end{cases} \qquad (8)$$

At $k\gg\gamma^{-1}$, Eq. (7) generates a stationary time series. The difference of the transit times t_{k+q} and t_k in Eq. (3), for $\tau_0=\bar{\tau}$ or $2\gamma k\gg 1$, is

$$t_{k+q}-t_k=\bar{\tau}q+\frac{\sigma}{\gamma}\left\{[1-(1-\gamma)^q]\sum_{l=1}^{k}(1-\gamma)^{k+1-l}\varepsilon_l+\sum_{l=k+1}^{k+q}[1-(1-\gamma)^{k+q+1-l}]\varepsilon_l\right\},\quad q\geqslant 0. \qquad (9)$$

The dispersion of this times difference equals

$$\langle(t_{k+q}-t_k)^2\rangle-\bar{\tau}^2q^2=\frac{\sigma^2}{2}g(q), \qquad (10)$$

where

$$g(q)=\frac{2}{\gamma^2}\left\{[1-(1-\gamma)^q]^2\sum_{l=1}^{k}(1-\gamma)^{2l}+\sum_{l=1}^{q}[1-(1-\gamma)^l]^2\right\},\quad q\geqslant 0. \tag{11}$$

Summation in Eq. (11) results in

$$g(q)=\frac{2}{\gamma^2}\left\{q-\frac{(1-\gamma)[1-(1-\gamma)^q]}{1-(1-\gamma)^2}\{2+[1-(1-\gamma)^q](1-\gamma)^{2k+1}\}\right\}. \tag{12}$$

At $\gamma q\ll 1$,

$$g(q)=\begin{cases}(2k+1)q^2+q/3+2q^3/3, & 2\gamma k\ll 1\\ \left(\frac{1}{\gamma}+\frac{1}{2}\right)q^2+\frac{1}{3}q-\frac{1}{3}q^3, & 2\gamma k\gg 1,\end{cases} \tag{12a}$$

while for $2\gamma k\gg 1$ we have

$$g(q)=\frac{2}{\gamma^2}\left[q-2\frac{(1-\gamma)[1-(1-\gamma)^q]}{1-(1-\gamma)^2}\right] \tag{13}$$

$$\simeq\begin{cases}\left(\frac{1}{\gamma}+\frac{1}{2}\right)q^2+\frac{1}{3}q-\frac{1}{3}q^3, & \gamma q\ll 1\\ \frac{2}{\gamma^2}\left(q+\frac{1}{2}\right)-\frac{2}{\gamma^3}+\cdots, & q\gg\gamma^{-1}\gg 1.\end{cases} \tag{13a}$$

Note that for $q<0$ we should replace q in Eqs. (9)–(13) by $|q|$ and k by $k-|q|$. Therefore, the function $g(q)$ at $k-|q|\gg\gamma^{-1}$ is even, i.e., $g(-q)=g(q)$.

Substituting Eq. (9) into Eq. (3), and replacing the summations in the exponents by the multiplications of the exponents, we have the following expression for the power spectral density of the current

$$S(f)=\lim_{T\to\infty}\left\langle\frac{2}{T}\sum_k\sum_{q=k_{\min}-k}^{k_{\max}-k}e^{i2\pi f\bar{\tau}q}\prod_{l=1}^{k}\exp\left\{i\frac{2\pi f\sigma}{\gamma}[1-(1-\gamma)^q](1-\gamma)^{k+1-l}\varepsilon_l\right\}\right.$$
$$\left.\times\prod_{l=k+1}^{k+q}\exp\left\{i\frac{2\pi f\sigma}{\gamma}[1-(1-\gamma)^{k+q+1-l}]\varepsilon_l\right\}\right\rangle. \tag{3$'$}$$

The average over realizations of the process coincides with the average over the distribution of the random variables ε_l. Using the fact that random variables ε_l are independent and mutually uncorrelated, we can fulfill the averaging over every random variable ε_l independently. Therefore, Eq. (3′) may be rewritten in the form

$$S(f)=\lim_{T\to\infty}\frac{2}{T}\sum_{k,q}e^{i2\pi f\bar{\tau}q}\prod_{l=1}^{k}\left\langle\exp\left\{i\frac{2\pi f\sigma}{\gamma}[1-(1-\gamma)^q](1-\gamma)^{k+1-l}\varepsilon_l\right\}\right\rangle\prod_{l=k+1}^{k+q}\left\langle\exp\left\{i\frac{2\pi f\sigma}{\gamma}[1-(1-\gamma)^{k+q+1-l}]\varepsilon_l\right\}\right\rangle. \tag{3$''$}$$

The result of the averaging of the exponent $\exp\{ic\varepsilon_l\}$ (with c being a constant) over the normally distributed random variable ε_l with zero expectation and unit variance is

$$\langle e^{ic\varepsilon_l}\rangle=\int_{-\infty}^{+\infty}e^{ic\varepsilon_l}\frac{1}{\sqrt{2\pi}}e^{-\varepsilon_l^2/2}d\varepsilon_l=e^{-c^2/2}.$$

Therefore, after the averaging over the normal distribution of the random variables ε_l, Eq. (3″) takes the form

$$S(f)=\lim_{T\to\infty}\frac{2}{T}\sum_{k,q}e^{i2\pi f\bar{\tau}q}\prod_{l=1}^{k}\exp\left\{-\frac{2\pi^2f^2\sigma^2}{\gamma^2}[1-(1-\gamma)^q]^2(1-\gamma)^{2(k+1-l)}\right\}\prod_{l=k+1}^{k+q}\exp\left\{-\frac{2\pi^2f^2\sigma^2}{\gamma^2}[1-(1-\gamma)^{k+q+1-l}]^2\right\}. \tag{3$'''$}$$

The transition in Eq. (3'''') from the multiplications of the exponents to the summations in the exponents and transformations, in analogy with Eq. (11) of the two sums' summation indexes $l \to k+1-l$ and $l \to k+q+1-l$, respectively, yields, according to Eq. (11), the final expression for the power spectral density:

$$S(f)=\lim_{T\to\infty}\frac{2}{T}\sum_{k,q} e^{i2\pi f\bar{\tau}q-\pi^2 f^2\sigma^2 g(q)}. \qquad (14)$$

Since the expansion of the function $g(q)$ in powers of $\gamma|q|\ll 1$ at $2\gamma k \gg 1$, according to Eqs. (12) and (13), is

$$g(q)=\frac{1}{\gamma}q^2-\frac{1}{3}|q|^3+\frac{1}{2}q^2\pm\cdots \qquad (15)$$

for $f\ll f_\tau=(2\pi\bar{\tau})^{-1}$ and $f<f_2=2\sqrt{\gamma}/\pi\sigma$, we can replace the summation in Eq. (14) by the integration

$$S(f)=2\bar{I}\int_{-\infty}^{+\infty} e^{i2\pi f\bar{\tau}q-\pi^2 f^2\sigma^2 g(q)}dq \qquad (16)$$

where $\bar{I}=\lim_{T\to\infty}(k_{\max}-k_{\min}+1)/T=(\bar{\tau})^{-1}$ is the averaged current.

Furthermore, at $f\gg f_1=\gamma^{3/2}/\pi\sigma$, it is sufficient to take into account only the first term of expansion (15), $g(q)=q^2/\gamma$. Integration in Eq. (16) hence yields the $1/f$ spectrum

$$S(f)=2\bar{I}\int_{-\infty}^{+\infty}\exp\left[i2\pi f\bar{\tau}q-\frac{(\pi f\sigma)^2}{\gamma}q^2\right]dq=\bar{I}^2\frac{\alpha_H}{f},$$

$$f_1<f<f_2,f_\tau \qquad (17)$$

where α_H is a dimensionless constant (the Hooge parameter)

$$\alpha_H=\frac{2}{\sqrt{\pi}}Ke^{-K^2}, \quad K=\frac{\bar{\tau}}{\sqrt{2}\sigma_\tau}=\frac{\bar{\tau}\sqrt{\gamma}}{\sigma}. \qquad (18)$$

Using an expansion of the function $g(q)$ at $\gamma q\gg 1$ according to expression (13), $g(q)=2q/\gamma^2$, from Eq. (16) we obtain the Lorentzian power spectrum density for $f<f_1$:

$$S(f)=2\bar{I}\frac{\sigma^2}{\bar{\tau}^2\gamma^2}\frac{1}{1+(\pi f\sigma^2/\bar{\tau}\gamma^2)^2}=\bar{I}^2\frac{4\tau_{\rm rel}}{1+\tau_{\rm rel}^2\omega^2}. \qquad (19)$$

Here $\omega=2\pi f$, and $\tau_{\rm rel}=D_t=\sigma^2/2\bar{\tau}\gamma^2$ is the "diffusion" coefficient of the time t_k according to Eqs. (7) and (8). The model is, therefore, free from the unphysical divergence of the spectrum at $f\to 0$; for $f\ll f_0=\bar{\tau}\gamma^2/\pi\sigma^2=1/2\pi\tau_{\rm rel}$, we have, from Eq. (19), the white noise

$$S(f)=\bar{I}^2(2\sigma^2/\bar{\tau}\gamma^2). \qquad (20)$$

Therefore, the model containing only one relaxation rate γ for a sufficiently small parameter γ can produce an exact $1/f$-like spectrum in any desirably wide range of frequency, $f_1<f<f_2,f_\tau$. Furthermore, due to the contribution to the transit times t_k of the large number of the very separated in time random variables, ε_l $(l=1,2,\ldots,k)$, our model represents a "long-memory" random process.

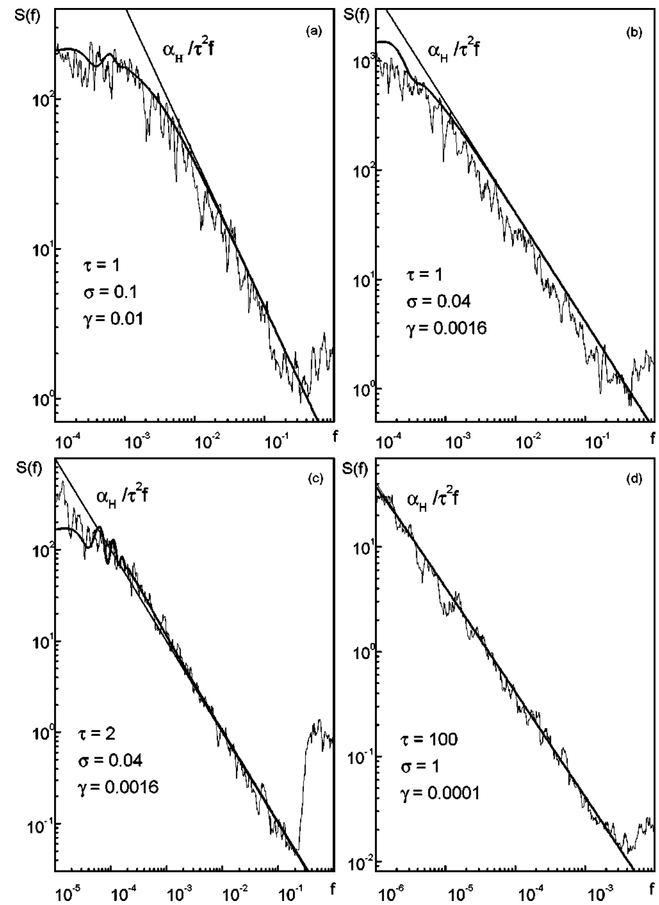

FIG. 1. Power spectral density vs frequency of the current generated by Eqs. (2)–(4) with the Gaussian distribution of the random increments $\{\varepsilon_k\}$ for different parameters $\bar{\tau}$, σ, and γ. The sinuous fine curves represent the results of numerical simulations averaged over five realizations, the heavy lines correspond to the numerical integration of Eq. (16) with $g(q)$ from Eq. (13), and the thin straight lines represent the analytical spectra according to Eqs. (17) and (18).

IV. GENERALIZATIONS AND NUMERICAL ANALYSIS

Equations (16)–(20) describe quite well the power spectrum of the random process (4). As an illustrative example, in Fig. 1 the numerically calculated power spectral density averaged over five realizations of process (4) is compared with the analytical calculations according to Eqs. (16)–(20). The analytical results are in good agreement with the numerical simulations. Note that analytical results predict not only the slope and intensity of $1/f$ noise, but the frequency range f_1–f_2,f_τ of $1/f$ noise and the intensity of the very low frequency $f\ll f_0$ white noise [Eq. (20)] as well.

This model may also be generalized for non-Gaussian and for continuous perturbations of the systems' parameters, resulting in the fluctuations of the period τ. So, for perturbations by the non-Gaussian sequence of random impacts $\{\varepsilon_k\}$ with zero expectations Eqs. (1)–(13) remain valid. Only result (14) of the averaging over realizations of the process in the case of the non-Gaussian perturbations may have a dif-

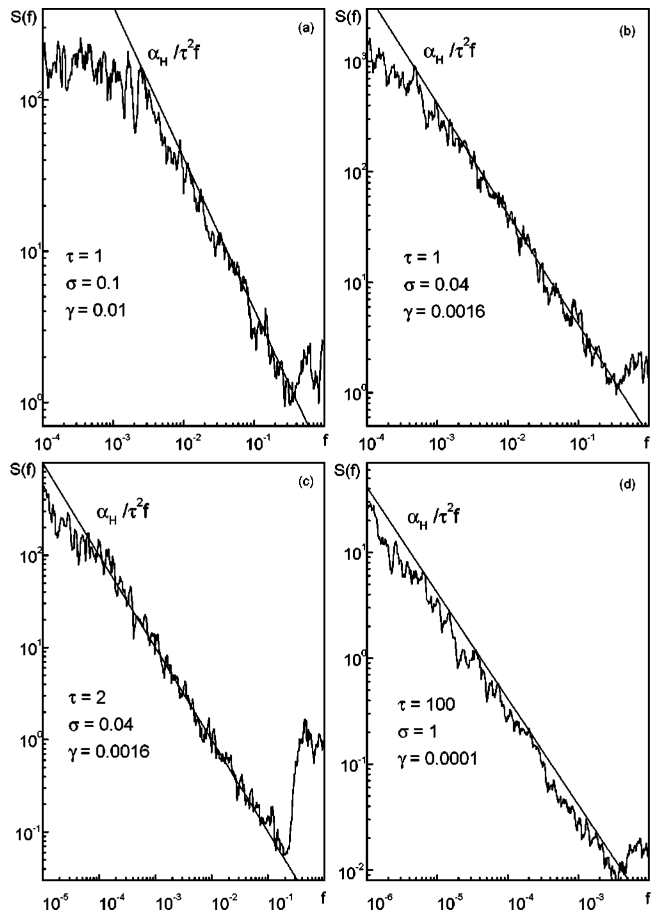

FIG. 2. Same as in Fig. 1 but for the uniform distribution of the random increments $\{\varepsilon_k\}$. Note that for the non-Gaussian distributions of the random perturbations we have no explicit expression analogous to Eq. (16) for the integral representation of the noise power spectral density.

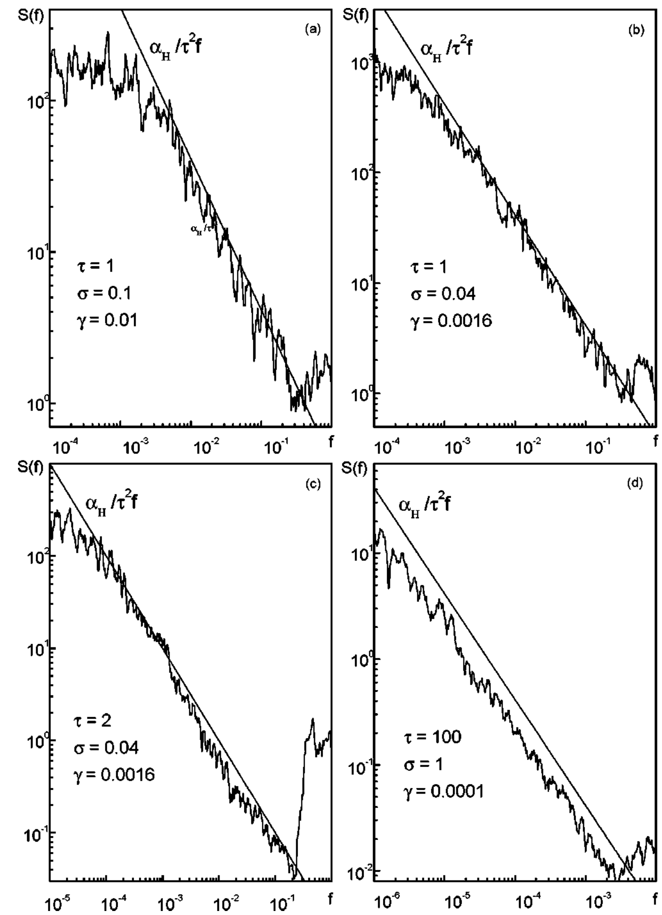

FIG. 3. Same as in Figs. 1 and 2, but for the asymmetric χ_3^2 distribution of the random increments $\{\varepsilon_k\}$.

ferent form. We now consider such a situation in more detail.

The power spectral density (3) may be rewritten in the form

$$S(f)=2\overline{I}\left\langle \sum_q e^{i2\pi f\tau_k(q)q}\right\rangle, \tag{21}$$

where the difference between the transit times t_{k+q} and t_k is expressed as

$$t_{k+q}-t_k=\sum_{l=k+1}^{k+q}\tau_k=\tau_k(q)q,\quad q\geqslant 0, \tag{22}$$

and the brackets denote the averaging over the time (index k) and over the realizations of the process. Here $\tau_k(q)\equiv(t_{k+q}-t_k)/q$ is the averaged time interval between the subsequent transit times in the time interval t_k–t_{k+q}. Note that for the slow (diffusivelike) fluctuations of the averaged interval $\tau_k(q)$ with the change of the index k, Eq. (22) is valid also when $q<0$, i.e., $t_{k+q}-t_k=\tau_{k+q}(q)q\simeq\tau_k(q)q$, $q<0$. At $2\pi f\tau_k(q)\ll 1$ we may replace the summation in Eq. (21) by the integration, and do not take into account the dependence of $\tau_k(q)$ on q. In such a case Eq. (21) yields

$$S(f)=2\overline{I}\left\langle\int_{-\infty}^{+\infty}e^{i2\pi f\tau_k q}dq\right\rangle=2\overline{I}\int_{-\infty}^{+\infty}\langle e^{i2\pi f\tau_k q}\rangle dq. \tag{23}$$

Here the averaging over k and over the realizations of the process coincides with the averaging over the distribution of the periods τ_k, i.e.,

$$\langle e^{i2\pi fq\tau_k}\rangle=\int_{-\infty}^{+\infty}e^{i2\pi fq\tau}\psi(\tau)d\tau=\chi_\tau(2\pi fq), \tag{24}$$

where $\psi(\tau)$ is the distribution density of periods τ_k, and $\chi_\tau(\vartheta)$ is the characteristic function of the distribution $\psi(\tau)$. Taking into account the property of the characteristic function,

$$\int_{-\infty}^{+\infty}\chi_\tau(\vartheta)d\vartheta=2\pi\psi(0),$$

we have, from Eqs. (23) and (24), the final expression for the power spectral density:

$$S(f)=2\overline{I}\psi(0)/f. \tag{25}$$

Substituting into Eq. (25) the value $\psi(0)=\exp(-\overline{\tau}^2/2\sigma_\tau^2)/\sqrt{2\pi}\sigma_\tau$ for the Gaussian distribution of the periods τ_k, we recover results (17) and (18).

Since different processes result in the Gaussian distribution, it is likely that perturbation by the non-Gausssian impacts $\{\varepsilon_k\}$ in Eq. (4) nevertheless yields the Gaussian distribution of the periods τ_k. For a demonstration of such a statement, and of the validity of approach (21)–(25), we have performed a numerical analysis of the model (1)–(4) for different distributions of the perturbations $\{\varepsilon_k\}$. Figures 2 and 3 represent the calculated power spectral densities for the rectangular (uniform) and asymmetric χ_3^2 distributions of the sequence $\{\varepsilon_k\}$, with zero expectations and the same variances and other parameters as those in Fig. 1. We note only the slight dependence of the spectra on the distribution function of the perturbing impacts $\{\varepsilon_k\}$, with the same expectations and variances. These results also confirm the presumptions made in the derivation (21)–(24) of the $1/f$ noise intensity [Eq. (25)].

V. CONCLUDING REMARKS

An analysis of the exactly solvable model of $1/f$ noise displays the main features of the noise, and serves as a basis for revealing the origin and parameter dependences of the flicker noise. This allows us to make generalizations of the model, resulting in an expression for the $1/f$ noise intensity through the integral of the characteristic function of the distribution of the time intervals between the subsequent transition times of the elementary events, pulses, or particles.

It should be noted, however, that Eq. (25) represents an idealized $1/f$ noise. Real systems have finite relaxation times, and, therefore, an expression of the noise intensity in the form of Eq. (23) is valid only for $f>(2\pi\tau_{\rm rel})^{-1}$, with $\tau_{\rm rel}$ being the relaxation time of the period's τ_k fluctuations. On the other hand, due to deviations from the approximation $t_{k+q}-t_k=\tau_k q$ at large q, for sufficiently low frequency we can obtain a finite intensity of $1/f^{\delta}$ $(\delta\simeq 1)$ noise even in the case $\psi(0)=0$, for signals with fluctuations resulting in dense concentrations of the transit times t_k. Generalizations of approach (21)–(25), and an analysis of the deviations from the idealized $1/f$ noise expression (25), are subjects of separate investigations.

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